Computing submatrices of the Hermite normal form of a structured polynomial matrix

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3 March 2026

Univariate polynomial matrices

Consider a univariate polynomial matrix M,

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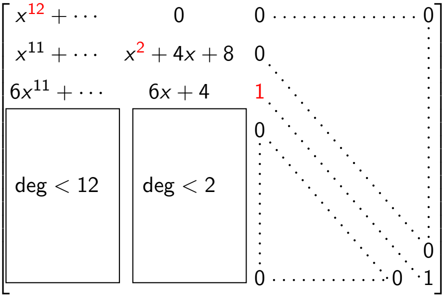
Hermite Normal Form

Maple, SageMath, ...

Univariate polynomial matrices



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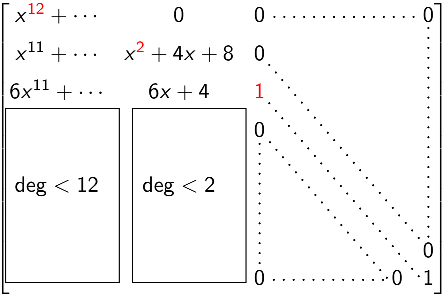

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[Gupta, Storjohann 2011]

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$\tilde{O}(n^{\omega}\,\deg(M))$

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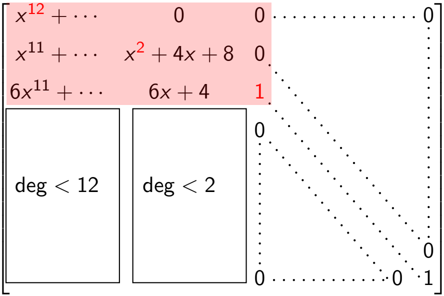

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In many applications

Determinant, Smith Form, Change of Ordering, ...

$\rightarrow 3 \times 3$ submatrix

Approach

Room for improvement?

Best known costs:
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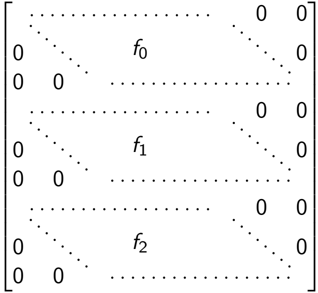
Structured matrices
Small displacement
rank $\alpha \ll n$



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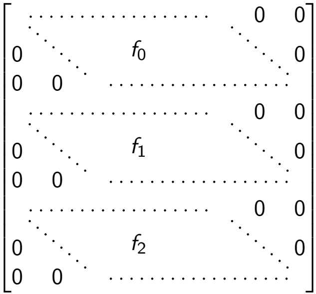

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[Bostan, Jeannerod, Mouilleron, Schost, 2017]

Linear algebra for structured matrices

over $\mathbb{K}^{n \times n}$ costs $\tilde{O}(\alpha^{\omega - 1} n)$

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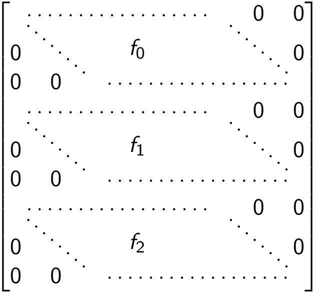

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For matrices over $\mathbb{K}[x]^{n\times n}$
$\rightarrow$ Evaluation interpolation

Contributions

[Berthomieu, Neiger, Passe 2026] Computing $m \times m$-submatrix of the HNF of an $n \times n$ matrix M with displacement rank α

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Generalizes the resultant by evaluation interpolation
$\alpha = 2, m = 1$

Application to bivariate change of ordering for Gröbner bases

$$\mathcal{G}_{\mathrm{DRL}} \longrightarrow \mathcal{G}_{\mathrm{LEX}}$$

Conclusions and perspectives

- Any Toeplitz-like matrix
- **Lower cost** bound in generic cases, detected **on the fly**
- Compute submatrix $H_{J,J}$ of the HNF, for $J = \{j_1, \cdots, j_m\}$
  - Leading submatrix, $J = \{1, \ldots, m\}$
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  - Fewer columns for relation basis: $\tilde{O}(m^{\omega-1} n \Delta) \to \tilde{O}(m^{\omega} \Delta)$?
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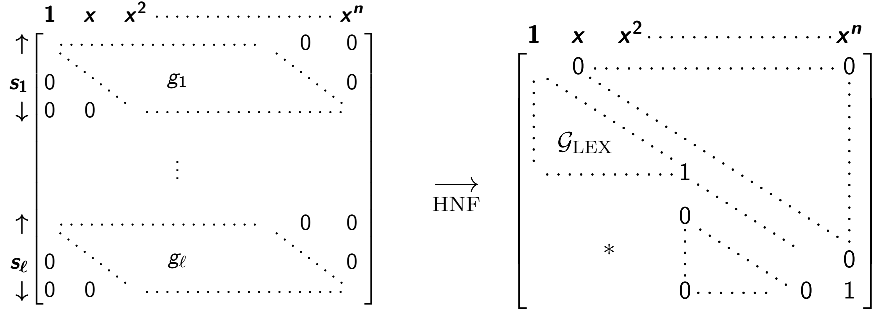
Thank you for your attention

Appendix

$\mathcal{G} = \{g_1, \ldots, g_\ell\}$ a DRL Gröbner basis of a bivariate ideal $I \subseteq \mathbb{K}[x, y]$.

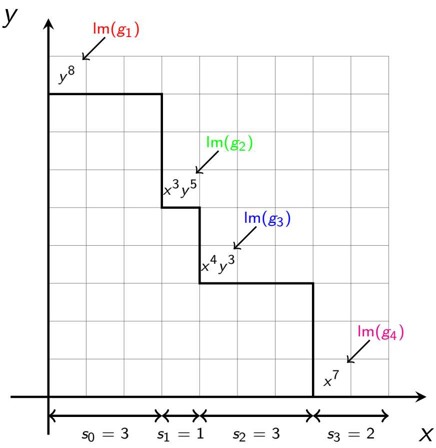


Univariate matrix in $\mathbb{K}[y]^{n \times n}$
Displacement rank ℓ

Upper-left submatrix
LEX Gröbner basis.

$$s_i = \deg_x \operatorname{lm}(g_{i+1}) - \deg_x \operatorname{lm}(g_i)$$

$$s_\ell = \max\left\{1, \max_{1 \le i \le \ell - 1}(s_i + \deg_x g_i) - \deg_x(g_\ell)\right\}$$



	1	x	x^2	x^3	x^4	x^5	x^6	x^7	x^8
g_1	$y^8 + \cdots$	$*$					$*$	0	0
xg_1	0	$y^8 + \cdots$							0
x^2g_1	0	0	$y^8 + \cdots$	$*$					$*$
g_2	$*$		$*$	$y^5 + \cdots$	$*$				$*$
g_3	$*$			$*$	$y^3 + \cdots$	$*$	$*$	0	0
xg_3	0					$y^3 + \cdots$	$*$	$*$	0
x^2g_3	0	0	$*$			$*$	$y^3 + \cdots$	$*$	$*$
g_4	$*$						$*$	1	0
xg_4	0	$*$						$*$	1

Matrix in $\mathbb{K}[y]^{9\times 9}$
Displacement rank 4.